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ON ESTIMATES FOR NORM OF SOME INTEGRAL OPERATORS IN WEIGHTED LEBESGUE SPACES

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Abstract. In this work, upper estimates for norm of an integral operator with Oinarov kernel in weighted Lebesgue spaces are obtained. The results given are very important in the study of the oscillation and non-oscillation properties of solutions of differential equations, as well as spectral properties.

1. Introduction

Let $(a,b) \subset R$ and u,v be weight functions in (a,b), i.e., positive measurable functions defined a.e. in (a,b). Let $p,q > 1$ and introduce weighted Lebesgue spaces

$$L_{p,v} = \{f : \|f\|_{p,v}^p := \int_a^b |f(t)|^p v(t)dt < \infty\}$$

and similarly $L_{q,u}$. In this paper we consider the integral operator

$$H : L_{p,v} \to L_{q,u},\ (Hf)(x) := \int_a^x k(x,t)f(t)dt, \tag{1.1}$$

where $k(x,t)$ is called kernel of the operator, which is nonnegative measurable function defined a.e. in $(a,b)\times(a,b)$.

The problem of boundedness of this operator in Lebesgue spaces began to be studied in the last decades of the last century. Let us now give some scientific conclusions concerning operators of this type. For example, F. J. Martin-Reyes and E. Sawyer [11] and V. D. Stepanov [14] considered the Riemann-Liouville fractional integral operator, i.e., (1.1) of the kernel $k(x,t) = \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}$, $\alpha \geqslant 1$ and $\Gamma(\alpha)$ is the Gamma function. S. Bloom and R. Kerman [2] and R. Oinarov [12, 13] gave equivalent conditions for boundedness of (1.1) for kernels $k(x,t)$ is a continuous nonnegative function increasing in the first argument, decreasing in the second argument and satisfying the condition: there exists a number $h \geqslant 1$ such that

$$k(x,s) \leqslant h(k(x,t)+k(t,s))$$

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for all $a < s \leqslant t \leqslant x < b$. Functions $k(x,t)$ satisfying the above conditions are also called *Oinarov's kernel*. In the works on this topic, the main focus was on finding equivalent conditions for the boundedness of the operator in (1.1), where estimates for the operator norm are very rare. There are some works in which the main purpose is to study another object, but the authors also gave estimates for the norm of the operator (1.1), see e.g. [3], [4], [6] and [7]. However, in the theory of differential equations it is very important to obtain the exact estimates.

Recently, in 2021 A. Kalybay and A. Baiarystanov [5] were obtained upper and lower estimates for the norm of the operator (1.1) in the case $1 < p \leqslant q < \infty$ as

$$A \leqslant \|H\|_{L_{p,v} \to L_{q,u}} \leqslant (h+1)^3 p^{\frac{1}{q}} (p')^{\frac{1}{p'}} A, \tag{1.2}$$

where $A = \max\{A_1, A_2\}$ and

$$A_1 = \sup_{a<x<b} \left(\int_x^b k^q(t,x) u(t) dt \right)^{\frac{1}{q}} \left(\int_a^x v^{1-p'}(t) dt \right)^{\frac{1}{p'}},$$

$$A_2 = \sup_{a<x<b} \left(\int_x^b u(t) dt \right)^{\frac{1}{q}} \left(\int_a^x k^{p'}(x,t) v^{1-p'}(t) dt \right)^{\frac{1}{p'}}.$$

But, the calculations show that for certain values of p and q it is possible to obtain a better estimate.

If we look at the right hand side of (1.2) we see two expressions $(h+1)^3$ and $p^{\frac{1}{q}} (p')^{\frac{1}{p'}}$, one related to h and the other to the powers p and q, i.e., a change in one expression does not affect the other. In this paper, we give another upper estimates for the operator norm, where in the expression these parameters are continuously dependent each other, which is useful to find a better estimate for the operator norm for some choice of p and q, see Example 2.3.

The paper is organized as follows. The first section is introduction. In the second section we give the main results. The proofs of the results are given in the third section.

2. Main results

The first result of this paper states that the upper estimate for the operator norm (2.1) can be given by a solution of some nonlinear equation. Unfortunately, such equations cannot always be solved analytically. However, now the time when computer technologies are advancing, such equations can be solved approximately.

The first result of the paper reads:

THEOREM 2.1. *Let $1 < p \leqslant q < \infty$. Then the norm of the integral operator in* (1.1) *satisfies*

$$A \leqslant \|H\| \leqslant C, \tag{2.1}$$

where C is a positive solution of the equation

$$C^{q'} - hq^{\frac{1}{q-1}} (q')^{\frac{1}{q-1}} A^{\frac{1}{q-1}} C = hq^{\frac{1}{q-1}} (p')^{\frac{q'}{p'}} A^{q'} \tag{2.2}$$

and $A = max\{A_1, A_2\}$.

REMARK 2.2. Equation (2.2) has a unique positive solution, since the function

$$h(x) = \frac{x^{q'}}{q^{\frac{1}{q-1}}(p')^{\frac{q'}{p'}}A^{q'} + q^{\frac{1}{q-1}}(q')^{\frac{1}{q-1}}A^{\frac{1}{q-1}}x}$$

is continuous and monotone increasing function of x in the half line $(0,\infty)$, $h(0) = 0$ and $h(\infty) = \infty$.

EXAMPLE 2.3. Let $1 < p \leqslant q = 2$ and $h \geqslant 1$. Then Equation (2.2) and it's positive solution take the forms

$$C^2 - 4AhC = 2h(p')^{\frac{2}{p'}}A^2$$

and

$$C = \left(2h + \sqrt{4h^2 + 2h(p')^{\frac{2}{p'}}}\right)A,$$

respectively. Then corresponding estimates take the form

$$A \leqslant \|H\| \leqslant \left(2h + \sqrt{4h^2 + 2h(p')^{\frac{2}{p'}}}\right)A.$$

REMARK 2.4. Similarly, the above results can be written for so-called conjugate Hardy operator

$$H^* : L_{p,v} \to L_{q,u}, \; (H^*f)(x) := \int_x^b k(t,x)g(t)dt \tag{2.3}$$

to the Hardy operator, i.e., the following estimates hold true for H^*:

$$A^* \leqslant \|H^*\| \leqslant C,$$

where C is a positive solution of (2.2) and $A^* = \max\{A_1^*, A_2^*\}$, where

$$A_1^* = \sup_{a<x<b}\left(\int_a^x k^p(x,t)u(t)dt\right)^{\frac{1}{p}}\left(\int_a^x v^{1-p'}(t)dt\right)^{\frac{1}{p'}},$$

$$A_2^* = \sup_{a<x<b}\left(\int_a^x u(t)dt\right)^{\frac{1}{p}}\left(\int_x^b k^{p'}(t,x)v^{1-p'}(t)dt\right)^{\frac{1}{p'}}.$$

Our next result reads as:

THEOREM 2.5. *Let* $1 < p \leqslant q < \infty$. *Then the norm of the integral operator in* (1.1) *satisfies*

$$A \leqslant \|H\| \leqslant eh^{q-1}qq'A, \tag{2.4}$$

where $A = max\{A_1, A_2\}$.

3. Proofs

In this section we present proofs of the main theorems.

Proof of Theorem 2.1. Here we first assume that the integral operator (1.1) is bounded, that is, $A<\infty$. Then it can be described in the form of this inequality

$$\left(\int_a^b\left(\int_a^x k(x,t)f(t)dt\right)^q u(x)dx\right)^{\frac{1}{q}} \leqslant C\left(\int_a^b f^p(x)v(x)dx\right)^{\frac{1}{p}},$$

which is so-called generalized Hardy inequality, where $C=\|H\|_{L_{p,v}\to L_{q,u}}$ is the best constant (the smallest constant in which the inequality holds) of the inequality.

We do not give a lower estimate for the norm, which is proved without change, as in Oinarov's theorem [9, Theorem 2.10] or in [5, Theorem 1]. Let denote

$$I=\int_a^b\left(\int_a^x k(x,t)f(t)dt\right)^q u(x)dx,$$

then consequently using Fubini's theorem and Hölder's inequality we get

$$\begin{aligned}
I&=q\int_a^b\left(\int_a^x k(x,t)f(t)\left(\int_a^t k(x,s)f(s)ds\right)^{q-1}dt\right)u(x)dx\\
&=q\int_a^b f(t)\left(\int_t^b k(x,t)u(x)\left(\int_a^t k(x,s)f(s)ds\right)^{q-1}dx\right)dt\\
&\leqslant q\|f\|_{p,v}\left(\int_a^b v^{1-p'}(t)\left(\int_t^b k(x,t)u(x)\left(\int_a^t k(x,s)f(s)ds\right)^{q-1}dx\right)^{p'}dt\right)^{1/p'}\\
&=q\|f\|_{p,v}J^{1/p'}. \qquad (3.1)
\end{aligned}$$

We now proceed to the proof by estimating J. To do this, we estimate its inner integral separately. Using Hölder's inequality with exponents $\frac{[q]}{q-1}$ and $\frac{[q]}{1-\{q\}}$ we have

$$\begin{aligned}
&\int_t^b k(x,t)u(x)\left(\int_a^t k(x,s)f(s)ds\right)^{q-1}dx\\
&=\int_t^b\left[k^{\{q\}}(x,t)u(x)\left(\int_a^t k(x,s)f(s)ds\right)^{[q]}\right]^{\frac{q-1}{[q]}}\times\left[k^q(x,t)u(x)\right]^{\frac{1-\{q\}}{[q]}}dx\\
&\leqslant\left(\int_t^b k^{\{q\}}(x,t)u(x)\left(\int_a^t k(x,s)f(s)ds\right)^{[q]}dx\right)^{\frac{q-1}{[q]}}\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1-\{q\}}{[q]}}.
\end{aligned}$$

We now derive the following estimate from the condition of the Oinarov kernel and then Newton's binomial formulae

$$\leqslant h^{q-1}\left(\int_t^b k^{\{q\}}(x,t)u(x)\left(k(x,t)\int_a^t f(s)ds+\int_a^t k(t,s)f(s)ds\right)^{[q]}dx\right)^{\frac{q-1}{[q]}}$$

$$\times\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1-\{q\}}{[q]}}$$

$$=h^{q-1}\left(\int_t^b k^{\{q\}}(x,t)u(x)\left(\sum_{n=0}^{[q]}C_{[q]}^n k^n(x,t)\left(\int_a^t f(s)ds\right)^n\right.\right.$$

$$\left.\left.\times\left(\int_a^t k(t,s)f(s)ds\right)^{[q]-n}\right)dx\right)^{\frac{q-1}{[q]}}\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1-\{q\}}{[q]}}$$

$$=h^{q-1}\left(\sum_{n=0}^{[q]}C_{[q]}^n\left(\int_t^b k^{\{q\}+n}(x,t)u(x)dx\right)\left(\int_a^t f(s)ds\right)^n\right.$$

$$\left.\times\left(\int_a^t k(t,s)f(s)ds\right)^{[q]-n}dx\right)^{\frac{q-1}{[q]}}\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1-\{q\}}{[q]}}. \tag{3.2}$$

Using the Hölder inequality to the first integral of the sum for $0<n<[q]$ we have

$$\int_t^b k^{\{q\}+n}(x,t)u(x)dx=\int_t^b\left(k^q(x,t)u(x)\right)^{\frac{n-1+\{q\}}{q-1}}\left(k(x,t)u(x)\right)^{\frac{[q]-n}{q-1}}dx$$

$$\leqslant\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{n-1+\{q\}}{q-1}}\left(\int_t^b k(x,t)u(x)dx\right)^{\frac{[q]-n}{q-1}}.$$

Then (3.2) is estimated as follows:

$$\leqslant h^{q-1}\left(\sum_{n=0}^{[q]}C_{[q]}^n\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{n-1+\{q\}}{q-1}}\left(\int_t^b k(x,t)u(x)dx\right)^{\frac{[q]-n}{q-1}}\left(\int_a^t f(s)ds\right)^n\right.$$

$$\left.\times\left(\int_a^t k(t,s)f(s)ds\right)^{[q]-n}dx\right)^{\frac{q-1}{[q]}}\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1-\{q\}}{[q]}}$$

$$=h^{q-1}\left(\sum_{n=0}^{[q]}C_{[q]}^n\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{n}{q-1}}\left(\int_t^b k(x,t)u(x)dx\right)^{\frac{[q]-n}{q-1}}\right.$$

$$\left.\times\left(\int_a^t f(s)ds\right)^n\left(\int_a^t k(t,s)f(s)ds\right)^{[q]-n}dx\right)^{\frac{q-1}{[q]}}$$

$$= h^{q-1}\left(\left(\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1}{q-1}}\left(\int_a^t f(s)ds\right)\right.\right.$$

$$\left.\left.+\left(\int_t^b k(x,t)u(x)dx\right)^{\frac{1}{q-1}}\left(\int_a^t k(t,s)f(s)ds\right)\right)\right)^{q-1}.$$

Therefore, we obtain

$$\int_t^b k(x,t)u(x)\left(\int_a^t k(x,s)f(s)ds\right)^{q-1}dx$$

$$\leqslant h^{q-1}\left[\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1}{q-1}}\left(\int_a^t f(s)ds\right)\right.$$

$$\left.+\left(\int_t^b k(x,t)u(x)dx\right)^{\frac{1}{q-1}}\left(\int_a^t k(t,s)f(s)ds\right)\right]^{q-1}.$$

From this we get

$$J=\int_a^b v^{1-p'}(t)\left(\int_t^b k(x,t)u(x)\left(\int_a^t k(x,s)f(s)ds\right)^{q-1}dx\right)^{p'}dt$$

$$\leqslant h^{(q-1)p'}\int_a^b v^{1-p'}(t)\left[\left(\int_t^b k^q(x,t)u(x)dx\right)^{\frac{1}{q-1}}\left(\int_a^t f(s)ds\right)\right.$$

$$\left.+\left(\int_t^b k(x,t)u(x)dx\right)^{\frac{1}{q-1}}\left(\int_a^t k(t,s)f(s)ds\right)\right]^{(q-1)p'}dt.$$

Using the Minkowski inequality, we obtain

$$\leqslant h^{(q-1)p'}\left(\left[\int_a^b v^{1-p'}(t)\left(\int_t^b k^q(x,t)u(x)dx\right)^{p'}\left(\int_a^t f(s)ds\right)^{(q-1)p'}dt\right]^{\frac{1}{(q-1)p'}}\right.$$

$$\left.+\left[\int_a^b v^{1-p'}(t)\left(\int_t^b k(x,t)u(x)dx\right)^{p'}\left(\int_a^t k(t,s)f(s)ds\right)^{(q-1)p'}dt\right]^{\frac{1}{(q-1)p'}}\right)^{(q-1)p'}$$

$$=h^{(q-1)p'}\left(I_1^{\frac{1}{(q-1)p'}}+I_2^{\frac{1}{(q-1)p'}}\right)^{(q-1)p'}.$$

where

$$I_1=\int_a^b v^{1-p'}(t)\left(\int_t^b k^q(x,t)u(x)dx\right)^{p'}\left(\int_a^t f(s)ds\right)^{(q-1)p'}dt,$$

$$I_2=\int_a^b v^{1-p'}(t)\left(\int_t^b k(x,t)u(x)dx\right)^{p'}\left(\int_a^t k(t,s)f(s)ds\right)^{(q-1)p'}dt.$$

From this and we have that

$$I\leqslant q\|f\|_{p,v}J^{1/p'}\leqslant q\|f\|_{p,v}h^{q-1}\left(I_1^{\frac{1}{(q-1)p'}}+I_2^{\frac{1}{(q-1)p'}}\right)^{q-1},$$

i.e.,

$$I^{\frac{1}{q-1}}\leqslant hq^{\frac{1}{q-1}}\|f\|_{p,v}^{\frac{1}{q-1}}\left(I_1^{\frac{1}{(q-1)p'}}+I_2^{\frac{1}{(q-1)p'}}\right).$$

Further estimate I_1 and I_2, separately. To estimate I_1, we use the following Hardy inequality

$$\begin{aligned}I_1^{\frac{1}{(q-1)p'}}&=\left(\int_a^b v^{1-p'}(t)\left(\int_t^b k^q(x,t)u(x)dx\right)^{p'}\left(\int_a^t f(s)ds\right)^{(q-1)p'}dt\right)^{\frac{1}{(q-1)p'}}\\&\leqslant C_{p,(q-1)p'}\left(\int_a^b f^p(t)v(t)dt\right)^{\frac{1}{p}},\end{aligned}\tag{3.3}$$

which holds, since its characteristic condition is satisfied

$$\begin{aligned}&\sup_{s\in(a,b)}\left(\int_s^b v^{1-p'}(t)\left(\int_t^b k^q(x,t)u(x)dx\right)^{p'}dt\right)^{\frac{1}{(q-1)p'}}\left(\int_a^s v^{1-p'}(t)dt\right)^{\frac{1}{p'}}\\&\leqslant\sup_{s\in(a,b)}\left(\int_s^b v^{1-p'}(t)\left(A_1^q\left(\int_a^t v^{1-p'}(x)dx\right)^{-\frac{q}{p'}}\right)^{p'}dt\right)^{\frac{1}{(q-1)p'}}\left(\int_a^s v^{1-p'}(t)dt\right)^{\frac{1}{p'}}\\&\leqslant A_1^{q'}\sup_{s\in(a,b)}\left(\int_s^b v^{1-p'}(t)\left(\int_a^t v^{1-p'}(x)dx\right)^{-q}dt\right)^{\frac{1}{(q-1)p'}}\left(\int_a^s v^{1-p'}(t)dt\right)^{\frac{1}{p'}}\\&\leqslant A_1^{q'}\left(\frac{1}{q-1}\right)^{\frac{1}{(q-1)p'}}<\infty.\end{aligned}$$

For the best constant $C_{p,(q-1)p'}$ in (3.3) the following estimate holds:

$$C_{p,(q-1)p'}\leqslant((q-1)p')^{\frac{1}{(q-1)p'}}(p')^{\frac{1}{p'}}A_1^{q'}\left(\frac{1}{q-1}\right)^{\frac{1}{(q-1)p'}}=(p')^{\frac{q'}{p'}}A_1^{q'}.$$

Then

$$I_1^{\frac{1}{(q-1)p'}}\leqslant(p')^{\frac{q'}{p'}}A_1^{q'}\|f\|_{p,v}.$$

Let us estimate I_2:

$$I_2=\int_a^b v^{1-p'}(t)\left(\int_t^b k(x,t)u(x)dx\right)^{p'}\left(\int_a^t k(t,s)f(s)ds\right)^{(q-1)p'}dt$$
$$=\int_a^b\left(\int_a^t k(t,s)f(s)ds\right)^{(q-1)p'}d\left(-\int_t^b\left(\int_s^b k(x,t)u(x)dx\right)^{p'}v^{1-p'}(s)ds\right)$$
$$=\int_a^b\left(\int_t^b\left(\int_s^b k(x,t)u(x)dx\right)^{p'}v^{1-p'}(s)ds\right)d\left(\int_a^t k(t,s)f(s)ds\right)^{(q-1)p'}$$

using the Minkowski inequality in the inner integral, we get

$$\leqslant\int_a^b\left(\int_t^b u(x)\left(\int_s^b k^{p'}(x,t)v^{1-p'}(s)ds\right)^{\frac{1}{p'}}dx\right)^{p'}d\left(\int_a^t k(t,s)f(s)ds\right)^{(q-1)p'}.$$

We estimate the inner integral of the last inequality in the form

$$\leqslant\int_a^b\left(\int_t^b u(x)A_2\left(\int_x^b u(s)ds\right)^{-\frac{1}{q}}dx\right)^{p'}d\left(\int_a^t k(t,s)f(s)ds\right)^{(q-1)p'}$$

$$=(q')^{p'}A_2^{p'}\int_a^b\left(\int_t^b u(x)dx\right)^{\frac{p'}{q'}}d\left(\int_a^t k(t,s)f(s)ds\right)^{(q-1)p'}$$

$$=(q')^{p'}A_2^{p'}\left(\left[\int_a^b\left(\int_t^b u(x)dx\right)^{\frac{p'}{q'}}d\left(\int_a^t k(t,s)f(s)ds\right)^{(q-1)p'}\right]^{\frac{q'}{p'}}\right)^{\frac{p'}{q'}}$$

$$\leqslant(q')^{p'}A_2^{p'}\left(\int_a^b u(x)\left(\int_a^x k(x,s)f(s)ds\right)^{q}dx\right)^{\frac{p'}{q'}}=(q')^{p'}A_2^{p'}I^{\frac{p'}{q'}}.$$

Then

$$I_2^{\frac{1}{(q-1)p'}}\leqslant(q')^{\frac{1}{q-1}}A_2^{\frac{1}{q-1}}I^{\frac{1}{q}}.$$

From the above estimates we have finally obtained this

$$I^{\frac{1}{q-1}}\leqslant hq^{\frac{1}{q-1}}\|f\|_{p,v}^{\frac{1}{q-1}}\left((p')^{\frac{q'}{p'}}A_1^{q'}\|f\|_{p,v}+(q')^{\frac{1}{q-1}}A_2^{\frac{1}{q-1}}I^{\frac{1}{q}}\right).$$

Using $I^{\frac{1}{q}}\leqslant\|H\|\|f\|_{p,v}$ we get

$$I^{\frac{1}{q-1}}\leqslant hq^{\frac{1}{q-1}}\|f\|_{p,v}^{\frac{q}{q-1}}\left((p')^{\frac{q'}{p'}}A_1^{q'}+(q')^{\frac{1}{q-1}}A_2^{\frac{1}{q-1}}\|H\|\right).$$

$$\left(\frac{I^{\frac{1}{q}}}{\|f\|_{p,v}}\right)^{q'} \leqslant hq^{\frac{1}{q-1}}\left((p')^{\frac{q'}{p'}}A_1^{q'}+(q')^{\frac{1}{q-1}}A_2^{\frac{1}{q-1}}\|H\|\right).$$

So we have the estimate for the best constant

$$\|H\|^{q'} \leqslant hq^{\frac{1}{q-1}}\left((p')^{\frac{q'}{p'}}A_1^{q'}+(q')^{\frac{1}{q-1}}A_2^{\frac{1}{q-1}}\|H\|\right),$$

i.e.

$$\|H\|^{q'}-hq^{\frac{1}{q-1}}(q')^{\frac{1}{q-1}}A^{\frac{1}{q-1}}\|H\| \leqslant hq^{\frac{1}{q-1}}(p')^{\frac{q'}{p'}}A^{q'}. \tag{3.4}$$

Consequently, we obtain

$$\frac{\|H\|^{q'}}{q^{\frac{1}{q-1}}(p')^{\frac{q'}{p'}}A^{q'}+q^{\frac{1}{q-1}}(q')^{\frac{1}{q-1}}A^{\frac{1}{q-1}}\|H\|} \leqslant h.$$

Let now consider the function

$$f(x)=\frac{x^{q'}}{q^{\frac{1}{q-1}}(p')^{\frac{q'}{p'}}A^{q'}+q^{\frac{1}{q-1}}(q')^{\frac{1}{q-1}}A^{\frac{1}{q-1}}x}$$

corresponding to the left hand side of the estimate. It is easy to see that this function is monotone increasing and continuous in $(0,\infty)$, $f(0)=0$ and $f(\infty)=\infty$, which implies that the equation has exactly one positive solution. If C is a solution of the equation, i.e.,

$$\frac{C^{q'}}{q^{\frac{1}{q-1}}(p')^{\frac{q'}{p'}}A^{q'}+q^{\frac{1}{q-1}}(q')^{\frac{1}{q-1}}A^{\frac{1}{q-1}}C}=h$$

then

$$\|H\| \leqslant C.$$

The proof is complete. □

Proof of Theorem 2.5. Substituting $C=xh^{q-1}qq'A$ in (2.2) we obtain the following equation

$$x^{q'}-x=\frac{(p')^{\frac{q'}{p'}}}{h^{q-1}q(q')^{q'}}. \tag{3.5}$$

It is easy to check that the equation has only one positive solution which belongs to $(1,\infty)$. We know that in general the equation can not be analytically solved, but from the other side it is enough to find an upper estimate to the solution. For this aim we consider the equation

$$y^{q'}-y=\frac{e^{\frac{q'}{e}}}{q(q')^{q'}}, \tag{3.6}$$

the right hand side of which is greater than the right side of (3.5), since $(p')^{\frac{1}{p'}} \leqslant e^{\frac{1}{e}}$ for all $p' > 1$. Then the corresponding solutions satisfy

$$x \leqslant y.$$

Further, we show that the solution of (3.6) lies in the interval $(1,e)$.

Let us consider the function

$$f(y) = y^{q'} - y - \frac{e^{\frac{q'}{e}}}{q(q')^{q'}} \tag{3.7}$$

for $y \in (1,\infty)$. It is known that $f(1) < 0$ and

$$f(e) = e^{q'} - e - \frac{e^{\frac{q'}{e}}(q'-1)}{(q')^{q'+1}} > 0. \tag{3.8}$$

The positivity of $f(e)$ follows from the properties of the function

$$\varphi(t) = e^t - e - \frac{e^{\frac{t}{e}}(t-1)}{t^{t+1}},$$

i.e., $\varphi(1) = 0$ and the function is monotone increasing. Therefore, $f(e) = \varphi(q') > 0$ for all $q' > 1$.

So, finally we have the existence of the zero point y of the function f in the interval $(1,e)$, which implies that the solution x of (3.5) is less than e.

The proof is complete. □

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